Real-Time Problem-Solving with Contract Algorithms

Correction of the Proofs of Theorems 2 and 4*

Theorem 2 *The minimal acceleration ratio needed to construct an interruptible algorithm from a given contract algorithm is* $r = 4$.

Proof: From Lemma 1 we know that for any sequence of contracts, $X = (x_1, x_2, ...)$, r must satisfy:

$$\forall i \geq 1 : \quad Q_{\mathcal{A}}(\frac{x_1 + x_2 + \cdots + x_{i+1}}{r}) \leq Q_{\mathcal{A}}(x_i)$$

From the strict monotonicity of $Q_{\mathcal{A}}$ we get:

$$\forall i \geq 1 : \quad \sum_{j=1}^{i+1} x_j \leq r x_i$$

Setting:

$$g_0 = 0 \quad \text{and} \quad g_i = \sum_{j=1}^{i} x_j, \; i \geq 1,$$

we can write the previous equation as:

$$\forall i \geq 1 : \quad g_{i+1} \leq r(g_i - g_{i-1}). \tag{4}$$

We know that the sequence $(g_i)_{i \geq 1}$ is an increasing sequence of positive numbers, so ρ, defined as follows:

$$\rho = \inf\{\frac{g_2}{g_1}, \frac{g_3}{g_2}, \cdots, \frac{g_{i+1}}{g_i}, \cdots\},$$

satisfies $\rho \geq 1$. From Equation (4) we obtain

$$\forall i \geq 1 : \quad g_i \geq g_{i-1} + \frac{g_{i+1}}{r},$$

and thus

$$\forall i \geq 2 : \quad \frac{g_i}{g_{i-1}} \geq 1 + \frac{g_{i+1}}{r g_{i-1}} \geq 1 + \frac{\rho^2}{r}.$$

Finally, we deduce that:

$$\rho \geq 1 + \frac{\rho^2}{r}$$

(so that $\rho > 1$) or that:

$$r \geq \frac{\rho^2}{\rho - 1} \geq 4,$$

as the function $\rho \longrightarrow \frac{\rho^2}{\rho-1}$ reaches a minimum of 4 on the interval $(1, \, +\infty)$ for $\rho = 2$. $\square$

*We are grateful to Reshef Meir for pointing out the error in the original proofs.

Theorem 4 *The minimal acceleration ratio needed to construct an interruptible algorithm to solve* m *problem instances with a given contract algorithm is* $r = (\frac{m+1}{m})^{m+1}$.

Proof: For any sequence of contracts, $X = (x_k)_{k \geq 1}$, r must satisfy:

$$\forall i \geq 1 : \quad Q_{\mathcal{A}}(\frac{x_1 + x_2 + ... + x_{i+m}}{mr}) \leq Q_{\mathcal{A}}(x_i)$$

From the strict monotonicity of $Q_{\mathcal{A}}$ we get:

$$\forall i \geq 1 : \quad \sum_{j=1}^{i+m} x_j \leq m r x_i$$

Setting:

$$g_0 = 0 \quad \text{and} \quad g_i = \sum_{j=1}^{i} x_j, \; i \geq 1,$$

we can write the previous equation as:

$$\forall i \geq 1 : \quad g_{i+m} \leq mr(g_i - g_{i-1}).$$

or

$$\forall i \geq 1 : \quad g_i \geq g_{i-1} + \frac{g_{i+m}}{mr}, \tag{9}$$

We know that the sequence $(g_i)_{i \geq 1}$ is an increasing sequence of positive numbers, so ρ, defined as follows:

$$\rho = \inf\{\frac{g_2}{g_1}, \frac{g_3}{g_2}, \cdots, \frac{g_{i+1}}{g_i}, \cdots\},$$

satisfies $\rho \geq 1$. From Equation (9) we obtain

$$\forall i \geq 2 : \quad \frac{g_i}{g_{i-1}} \geq 1 + \frac{g_{i+m}}{mr g_{i-1}} \geq 1 + \frac{\rho^{m+1}}{mr}.$$

Finally, we deduce that:

$$\rho \geq 1 + \frac{\rho^{m+1}}{mr}$$

(so that $\rho > 1$) or that:

$$r \geq \frac{\rho^{m+1}}{m(\rho - 1)}.$$

The function $\rho \longrightarrow \frac{\rho^{m+1}}{\rho-1}$ reaches its minimum on the interval $(1, +\infty)$ when $\rho = \frac{m+1}{m}$, therefore $r \geq (\frac{m+1}{m})^{m+1}$.

The ratio $(\frac{m+1}{m})^{m+1}$ can be obtained by a sequence of contracts defined by a geometric series with run-times being multiplied by a factor of $\frac{m+1}{m}$. Thus the best possible acceleration ratio is $r = (\frac{m+1}{m})^{m+1}$. $\square$